Invertible Mapping

Let V and W be vector space over a field F. A linear mapping $T: V \to W$ is said to be invertible if $\exists$ a mapping $S: W \to V$ s.t. $ST = I_V$ and $TS = I_W$ where I_V and I_W are the identity operators in V and W respectively. In this case S is said to be an inverse of T and is denoted by T^{-1}.

[Here ST and TR are respectively the composite mapping $S \circ T$ and $T \circ R$ where

$$S \circ T = ST : V \to V \text{ and}$$
$$T \circ S = TS : W \to W.$$

If S & T are linear mapping then it can be prove that composite mapping is also linear.]

Theorem.

Let V and W be vector space over the same field F. If $T: V \to W$ be invertible then T has an unique inverse.

Proof.

If possible, let there be two inverses:
$T_1 : W \to V$ and $T_2 : W \to V$.

Then by definition,

$$T_1 T = T_2 T = I_V \text{ \& } T T_1 = T T_2 = I_W$$

Now by associative law,

$$T_1 (T T_2) = (T_1 T) T_2 \Rightarrow T_1 I_W = I_V T_2 \Rightarrow T_1 = T_2$$

This proves that inverse of T is unique.

Theorem:

Let V and W be vector space over the same field F. A linear mapping $T: V \to W$ is invertible iff T is one-to-one and onto.

Proof: First let $T: V \to W$ is invertible. Then $\exists$ a mapping $S: W \to V$ such that

$ST: V \to V = I_V$ & $TS: W \to W = I_W$.

Let $\alpha, \beta \in V$ s.t $T(\alpha) = T(\beta)$

$\Rightarrow ST(\alpha) = ST(\beta)$

$\Rightarrow I_V(\alpha) = I_V(\beta)$ since $ST = I_V$.

$\Rightarrow \alpha = \beta$.

$\therefore$ T is one-to-one.

Again let $\gamma \in W$.

As $TS = I_W$ so $TS(\gamma) = \gamma$.

i.e, $T(S(\gamma)) = \gamma$

which shows that $S(\gamma)$ is a pre-image of γ under T.

Therefore T is onto.

Thus we proved that if T is invertible then T is one-to-one as well as T is onto.

Next let T is one-to-one & onto.

We shall prove that T is invertible.

For that let $\alpha \in V$ s.that $T(\alpha) = \beta \in W$.

i.e. β is the unique image of α (as T is one-to-one)

Since T is onto, each $\beta \in W$ has a pre-image in V. Let us define $S: W \to V$ by $S(\beta) = \alpha$

Then $ST(\alpha) = S(\beta) = \alpha$, $\forall \alpha \in V$.

& $TS(\beta) = T(\alpha) = \beta$, $\forall \beta \in W$.

$\Rightarrow ST = I_V$ & $TS = I_W$

Hence T is invertible.

~~Art~~ Theorem

Let V and W be vector space over the same field F. If a linear mapping $T: V \to W$ is invertible, then the inverse mapping $T^{-1}: W \to V$ is linear.

Proof.

Here $T^{-1}: W \to V$ is the inverse mapping of the linear mapping $T: V \to W$ so that $TT^{-1} = I_W$ and $T^{-1}T = I_V$.

Let $\gamma, \delta \in W$ be such that

$T^{-1}(\gamma) = \alpha \in V$ & $T^{-1}(\delta) = \beta \in V$.

Thus $T(\alpha) = \gamma$ & $T(\beta) = \delta$.

Since T is linear, we have,

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad a, b \in F$$

$$\Rightarrow T(a\alpha + b\beta) = a\gamma + b\delta$$

$$\Rightarrow T^{-1}[T(a\alpha + b\beta)] = T^{-1}(a\gamma + b\delta)$$

$$\Rightarrow I_V(a\alpha + b\beta) = T^{-1}(a\gamma + b\delta)$$

$$\Rightarrow (a\alpha + b\beta) = T^{-1}(a\gamma + b\delta)$$

$$\Rightarrow aT^{-1}(\gamma) + bT^{-1}(\delta) = T^{-1}(a\gamma + b\delta)$$

This proves that T^{-1} is linear.

Definition.

A linear mapping $T: V \to W$ is said to be non-singular if T be invertible.

Ex: Let S & T be linear mapping of $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $S(x,y,z) = (z,y,x)$

and $T(x,y,z) = (x+y+z, y+z, z)$.

i) Determine TS and ST

ii) Prove that both S & T are invertible.

iii) Verify that $(ST)^{-1} = T^{-1}S^{-1}$

Sol: Using definition, since $S(x,y,z) = (z,y,x)$

$\therefore TS(x,y,z) = T(S(x,y,z)) = T(z,y,x)$

$= (z+y+x, y+x, x)$

& $ST(x,y,z) = S(T(x,y,z))$

$= S(x+y+z, y+z, z)$

$= (z, y+z, x+y+z)$.

(ii) Now $\ker TS$ & $\ker ST = \{0\}$ so that

$TS = ST = I_{\mathbb{R}^3}$.

Hence both S & T are invertible.

iii) Now, $ST(x,y,z) = (z, y+z, x+y+z) = (a,b,c)$

$\therefore \cancel{(ST)^{-1}(z, y+z, x+y+z)} = (ST)^{-1}(a,b,c)$

$\therefore a = z,\ b = y+z \Rightarrow y = b-a,\ c = x+y+z$

$\Rightarrow x = c-b+a-a$

$= c-b$.

$\therefore (ST)^{-1}(a,b,c) = (x,y,z) = (c-b, b-a, a)$

$\therefore (ST)^{-1}(x,y,z) = (z-y, y-x, x)$

Now $T(x,y,z) = (x+y+z, y+z, z) = (a_1, b_1, c_1)$

$\therefore x+y+z = a_1,\ y+z = b_1,\ z = c_1$

$\Rightarrow x = a_1 - b_1,\ y = b_1 - c_1$

$\therefore T^{-1}(a_1, b_1, c_1) = (x,y,z) = (a_1-b_1, b_1-c_1, c_1)$.

$\therefore T^{-1}(x,y,z) = (x-y, y-z, z)$.

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Again $S(x,y,z) = (z,y,x) = (a_2, b_2, c_2)$.

$\therefore$ $a_2 = z,\ b_2 = y,\ c_2 = x$.

$\therefore S^{-1}(a_2, b_2, c_2) = (x,y,z) = (c_2, b_2, a_2)$

$\therefore S^{-1}(x,y,z) = (z, y, x)$

Hence $T^{-1}S^{-1}$ is given by

$$T^{-1}S^{-1}(x,y,z) = T^{-1}(S^{-1}(x,y,z))$$
$$= T^{-1}(z,y,x)$$
$$= (z-y,\ y-x,\ x).$$

Hence $(ST)^{-1} = T^{-1}S^{-1}$.

Prob.

Let T be a linear operator on R^3 defined by $T(x,y,z) = (2x, 4x-y, 2x+3y-z)$. Show that T is invertible and find a formula for T^{-1}.

Solⁿ. T is a linear mapping. Let us find $\ker T$.

Let $(a,b,c) \in \ker T$.

Then $(2a,\ 4a-b,\ 2a+3b-c) = (0,0,0)$

Therefore $a = 0,\ b = 0,\ c = 0$.

$\therefore \ker T = \{0\}$ & hence ~~Ker~~ T is one-to one.

Again $V = R^3$ & $W = R^3$ so $\dim V = \dim W$.

Since T is one-to-one so T is onto.

Hence T is non-singular.

Let $T^{-1}(x,y,z) = (a,b,c) \Rightarrow T(a,b,c) = (x,y,z)$

$\therefore (2a,\ 4a-b,\ 2a+3b-c) = (x,y,z)$

$\therefore x = 2a \Rightarrow a = x/2$, $4a - b = y \Rightarrow b = 2x - y$, $2a+3b-c = z$

$\therefore c = x + 3(2x-y) - z$
$= 7x - 3y - z$

$\therefore T^{-1}(x,y,z) = \left(\frac{x}{2},\ 2x-y,\ 7x-3y-z\right)$.